- Consider the pendulum equation with a small angle approximation without forcing:
  - $\theta'' + \frac{g}{L}\sin\theta = 0$; $\sin\theta \approx \theta$; $\theta'' + \frac{g}{L}\theta = 0$; $\theta(t) = c_1 \cos\sqrt{\frac{g}{L}}t + c_2 \sin\sqrt{\frac{g}{L}}t$; $\omega_o = \sqrt{\frac{g}{L}}$
  - Hence, represent this equation generally as $x'' + \omega_o^{\,2} x = 0$
- General equation with forcing $x'' + \omega_o^{\,2} x = f(t)$, where $f(t)$ has angular frequency ω.
  - The associated frequency of ω_o is $f = \frac{2\pi}{\omega_o}$. This is the **natural frequency**.
  - Suppose that $f(t)$ is a sinusoidal function. Let $f(t) = \mathrm{Re}[Ce^{i\omega t}] + \mathrm{Im}[Ce^{i\omega t}]$
    - Note that $f(t)$ still has angular frequency ω.
- **Beats** ($\omega \neq \omega_o$)
  - $x_p = \frac{e^{\alpha t}}{p(\alpha)}$ $\bar{x}_p = \frac{Ce^{i\omega t}}{p(i\omega)}$ $\bar{x}_p = \frac{Ce^{i\omega t}}{\omega_o^{\,2} - \omega^2}$ $x_p = \frac{C}{\omega_o^{\,2} - \omega^2}(\cos\omega t + \sin\omega t)$
  - $x_p = \mathrm{Re}[\bar{x}_p] + \mathrm{Im}[\bar{x}_p]$ $x_c = c_1 \cos\omega_o t + c_2 \sin\omega_o t$
  - $x = c_1 \cos\omega_o t + c_2 \sin\omega_o t + \frac{C}{\omega_o^{\,2} - \omega^2}(\cos\omega t + \sin\omega t)$
  - For simplicity, $x'(0) = x(0) = 0$. $x = \frac{C}{\omega_o^{\,2} - \omega^2}(\cos\omega t - \cos\omega_o t + \sin\omega t - \sin\omega_o t)$
  - $x = \frac{2C}{\omega_o^{\,2} - \omega^2}\sin\left(\frac{\omega - \omega_o}{2}t\right)\left(\cos\left(\frac{\omega_o + \omega}{2}t\right) - \sin\left(\frac{\omega_o + \omega}{2}t\right)\right)$ Use trig identities.
  - $\sin\left(\frac{\omega - \omega_o}{2}t\right)$ represents the beats. Period $T = \frac{2\pi}{\omega - \omega_o}$.
  - $\cos\left(\frac{\omega_o + \omega}{2}t\right) - \sin\left(\frac{\omega_o + \omega}{2}t\right)$ represents the rapid oscillations. Period $T = \frac{4\pi}{\omega + \omega_o}$
- **Resonance**
  - A phenomenon that occurs with second-order linear non-homogeneous ODEs when $\omega = \omega_o$ (or $\omega = \sqrt{\omega_o^{\,2} - 2p^2}$ with damping given $x'' + 2px' + \omega_o^{\,2} x = f(t)$).
  - The solution "blows up" – amplitude escapes to infinity
  - Examples: child on a swing, Tacoma Narrows Bridge
  - $x_p = \frac{txe^{\alpha t}}{p'(\alpha)}$ $\bar{x}_p = \frac{Cte^{i\omega_o t}}{p'(i\omega_o)}$ $\bar{x}_p = \frac{Cte^{i\omega_o t}}{2i\omega_o}$ $\bar{x}_p = -\frac{Cite^{i\omega_o t}}{2\omega_o}$ $x_p = \mathrm{Re}[\bar{x}_p] + \mathrm{Im}[\bar{x}_p]$
  - $x_p = \frac{Ct}{2\omega_o}(\sin\omega_o t - \cos\omega_o t)$ $x = c_1 \cos\omega_o t + c_2 \sin\omega_o t + \frac{Ct}{2\omega_o}(\sin\omega_o t - \cos\omega_o t)$

    Notice that the amplitude of x_p increases linearly. This is resonance.
  - Note: resonance still occurs with other non-sinusoidal periodic functions $f(t)$ with angular frequency ω_o, such as square waves, triangle waves, and sawtooth waves.